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# On The Genus of Extended Dot Product Graph and The Crosscap of Dot Product Graph

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## Abstract

Let $S = C \times \cdots \times C$ ($k$ times) and $1 \leq k < \infty$ be an integer, where $C$ is a commutative ring with unity $1 \neq 0$. In this paper, we classify the rings $S$ such that the extended total dot product graph $\overline{TD}(S)$ and the extended zero-divisor dot product graph $\overline{ZD}(S)$ have genus one and two. Furthermore, we characterize $S$ such that the total dot product graph $TD(S)$ and the zero-divisor dot product graph $ZD(S)$ have crosscap one and two.

**Keywords:** dot product graph; genus; crosscap; reduced ring.

## 1 Introduction

The exploration of algebraic graph assignments to commutative rings began with Beck's work [10], which primarily focused on graph coloring. In 1999, Anderson and Livingston [7] studied ring theory in relation to graph theory by introducing zero-divisor graph $\Gamma(S)$, where $S$ is a commutative ring with unity. The vertex set of this graph consists of $Z(S)^*$ (nonzero zero-divisors), and two distinct vertices $r, s \in Z(S)^*$ form an edge if and only if $rs = 0$. Anderson and Livingston studied the connectedness, diameter and girth of $\Gamma(S)$. Further this work is continued by Akbari and Mohammadian [2, 3]. Since then, investigations into the characteristics of a ring through its zero-divisor graph have developed into a rich field of study, marked by continuous inquiries and valuable insights [6]. The extended zero-divisor graph, denoted by $\overline{\Gamma}(S)$, was introduced by Bennis et al. [11], in which the vertex set of $\overline{\Gamma}(S)$ is taken as $Z(S)^*$. The adjacency between $w$ and $z$ from $Z(S)^*$ is contingent upon the condition $w^k z^\ell = 0$, where $w^k \neq 0$ and $z^\ell \neq 0$, where $k$ and $\ell$ are positive integers. There are various other extensions of the zero-divisor graph has been done two of them are studied [4, 13].

Let $S = C \times \cdots \times C$ ($k$ times) and $1 \leq k < \infty$ be an integer, where $C$ is a commutative ring with $1 \neq 0$. In 2015, Badawi [8] introduced the idea of dot product graph. The author defined total dot product graph $TD(S)$ with vertex set $S^* = S \setminus \{(0, 0, \ldots, 0)\}$ and zero-divisor dot product graph $ZD(S)$ with vertex set $Z(S)^*$. Two vertices are taken from $S^*$ and $Z(S)^*$, respectively and two distinct vertices $t = (t_1, t_2, \ldots, t_m)$ and $s = (s_1, s_2, \ldots, s_m)$ are adjacent if and only if $t \cdot s = 0$. Badawi [8] proved some results regarding the connectedness, diameter and girth of this graph. In 2020, Selvakumar et al. [14] further explored these graphs, categorized $S$, for which $TD(S)$ and $ZD(S)$ have a genus of zero, one, or two. Also there are work related to topological properties of zero-divisor graph and its extended zero-divisor graph are studied in [16, 15].

In 2024, Asma et al. [5] extended the dot product graph and defined two new types of graphs: the extended total dot product graph $\overline{TD}(S)$, with the set $S^*$ as the vertex set, and the extended zero-divisor dot product graph $\overline{ZD}(S)$, with the set $Z(S)^*$ as the vertex set. Two vertices $w$ and $z$ (distinct) are adjacent if and only if $w^k \cdot z^\ell = 0$, where $w^k \neq 0$ and $z^\ell \neq 0$. The main objective of [5] was to examine the basic properties of these graphs and also classify the rings $S$ for which $\overline{TD}(S)$ and $\overline{ZD}(S)$ are planar/outerplanar. This paper aims to determine the if and only if conditions under which these graphs have a genus one or two, and also classify the ring $S$ such that $TD(S)$ and $ZD(S)$ have a crosscap number of one or two.

Throughout the paper, we consider $C$ to be a commutative ring with unity $1 \neq 0$, and $S = C \times C \times \cdots \times C$ ($k$ times), where $k$ is a positive integer. We also consider $S^\times$ as the set of units of $S$ and $C^\times$ as the set of units of $C$. A ring $S$ with no nonzero nilpotent element is called reduced. Let $H$ be a graph with vertex set $V$ and edge set $E$. A graph is said to be connected if there is a path between every pair of distinct vertices in $H$; if not, it is termed disconnected. A complete bipartite graph $K_{|V_1|,|V_2|}$ splits the vertex set $V$ into two subsets, $V_1$ and $V_2$. In this graph, each vertex in $V_1$ is adjacent to every vertex in $V_2$, but no two distinct vertices within the same subset are adjacent, where $|V_1|$ and $|V_2|$ denote the cardinalities of the sets $V_1$ and $V_2$, respectively. A planar graph is one that can be embedded on a plane such that its edges meet only at their endpoints.

Let $\mathbb{S}_n$ represent a sphere with $n$ handles, where $n$ is a natural number; in other words, $\mathbb{S}_n$ is an orientable surface characterized by having $n$ handles. The minimum value of $n$ that permits a graph $H$ to be embedded in the orientable surface in $\mathbb{S}_n$ is known as the genus of the graph, denoted by $\gamma(H)$. In other words, the smallest number of handles needed for its embedding on an orientable surface is called the genus of a graph. A graph is termed as planar if $\gamma(H) = 0$. If $\gamma(H) = 1$, then, the graph is referred as toroidal and a graph having genus two is known as bi-toriodal. The crosscap number $\overline{\gamma}(H)$ is the minimum number of projective planes $k$ required to embed $H$ in $\overline{\mathbb{S}}_n$, where $\overline{\mathbb{S}}_n$ denote a sphere with $k$ projective planes. If a graph $H$ that can be embedded into $\overline{\mathbb{S}}_1$ but cannot be embedded into a plane, then, it is called crosscap one of $H$ and if a graph that can be embedded into $\overline{\mathbb{S}}_2$ but not in $\overline{\mathbb{S}}_1$, then, it is called crosscap two of $H$. The minor subgraph $H'$ of a graph $H$ is a graph obtained through the contraction of its edges and isolation of vertices. The contracted edges of a minor subgraph are symbolized by the vertex $[u, z]$, where $u$ and $z$ belong to the vertex set $V(H)$. If the graph $H$ have a minor subgraph $H'$, then, the genus and crosscap of $H'$ is less than or equal to the genus and crosscap of $H$.

## 2 Genus of $\overline{TD}(S)$ and $\overline{ZD}(S)$

In this section, we categorize the rings $S = C \times C \times \cdots \times C$ ($k$ times) such that $\overline{TD}(S)$ and $\overline{ZD}(S)$ are of genus one and two.

**Lemma 2.1.** [17, *Theorem 6.37*] *If $r, s \geq 2$ are integers, then,*

$$\gamma(K_{r,s}) = \left\lceil \frac{(r-2)(s-2)}{4} \right\rceil.$$

**Lemma 2.2.** [14, *Lemma 4.2*] *If $H$ be a simple connected graph with $n$ vertices, $m$ edges and girth $gr$, then,*

$$\gamma(H) \geq \frac{m(gr-2)}{2gr} - \frac{n}{2} + 1.$$

**Lemma 2.3.** [9, *Theorem 4.3*] *The genus of a graph is the total of the genera of its individual blocks.*

**Theorem 2.1.** [5, *Theorem 4.1*] *Let $S = C \times C \times \cdots \times C$ ($k$ times), where $2 \leq k < \infty$. Then, $\overline{ZD}(S)$ is planar if and only if $S \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.*

**Theorem 2.2.** [5, *Theorem 4.2*] *Let $S = C \times C \times \cdots \times C$ ($k$ times), where $2 \leq k < \infty$. Then, $\overline{TD}(S)$ is planar if and only if $S \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.*

**Lemma 2.4.** *If $C \cong \mathbb{Z}_2$ and $S = C \times C \times \cdots \times C$ ($k$ times), where $4 \leq k < \infty$, then, $\gamma(\overline{ZD}(S)) \geq 3$.*

*Proof.* Suppose $k \geq 4$, and consider the set $H = H_1 \cup H_2$, where

$$\begin{aligned}
H_1 = \Big\{ & h_1 = (1,0,0,1,0,\ldots,0), h_2 = (0,1,1,0,\ldots,0), h_3 = (1,0,1,0,\ldots,0), \\
& h_4 = (0,1,0,1,\ldots,0), h_5 = (1,1,0,0,\ldots,0), h_6 = (0,0,1,1,0,\ldots,0) \Big\}, \\
H_2 = \Big\{ & h'_1 = (0,1,0,0,\ldots,0), h'_2 = (1,1,1,0,\ldots,0), h'_3 = (0,0,1,0,\ldots,0), \\
& h'_4 = (1,1,0,1,0,\ldots,0), h'_5 = (0,0,0,1,0,\ldots,0), h'_6 = (1,1,0,1,\ldots,0), \\
& h'_7 = (1,0,0,\ldots,0), h'_8 = (0,1,1,1,0,\ldots,0) \Big\}.
\end{aligned}$$

Then, $H$ is the smallest set of vertices contained in $V(\overline{ZD}(S))$. On contracting the vertices $[h_1, h_2]$, $[h_3, h_4]$ and $[h_5, h_6]$, we get the graph which is the minor subgraph of $\overline{ZD}(S)$, shown in Figure 1. Since we observe that the minor subgraph has eleven vertices and forty edges, so, by Lemma 2.2, we get $\gamma(\overline{ZD}(S)) \geq 3$. □

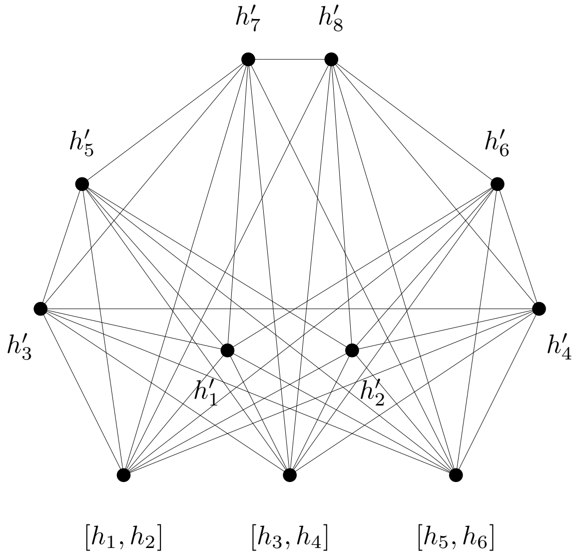


Figure 1: Minor subgraph of $\overline{ZD}(S)$.

Selvakumar et al. [14, Theorem 5.1] proved that $\gamma(ZD(C \times C \times C)) \geq 3$, if $|C| \geq 3$. Since $\overline{ZD}(S)$ span $ZD(S)$ (spanning subgraph of $\overline{ZD}(S)$), then, $\gamma(\overline{ZD}(C \times C \times C)) \geq 3$. Hence, we deduce the following:

**Lemma 2.5.** *If* $|C| \geq 3$, *and* $S = C \times C \times C$. *Then,* $\gamma(\overline{ZD}(S)) \geq 3$.

The lemma can also be extended as follows:

**Lemma 2.6.** *Let* $S = C \times C \times \cdots \times C$ *(k times), where* $2 \leq k < \infty$ *and* $C$ *is a nonreduced ring. Then,* $\gamma(\overline{ZD}(S)) \geq 3$.

*Proof.* Since $C$ is nonreduced, then, $|C| \geq 4$. Let $z \in N(C)^*$ (nonzero nilpotent elements of $C$) and $1 \neq k \in C^\times$ (units of $C$). Consider the set $D = \{r_1, r_2, r_3, r_4, r_5, s_1, s_2, s_3, s_4, s_5\}$, where,

$$r_1 = (1, 0, ..., 0), \quad r_2 = (z, 0, ..., 0), \quad r_3 = (k, 0, ..., 0), \quad r_4 = (1, z, 0, ..., 0), \quad r_5 = (k, z, 0, ..., 0),$$
$$s_1 = (0, 1, 0, ..., 0), \quad s_2 = (0, z, 0, ..., 0), \quad s_3 = (0, k, 0, ..., 0), \quad s_4 = (z, 1, 0, ..., 0), \quad s_5 = (z, k, 0, ..., 0).$$

Then, the graph obtained from the set $D$ contains the induced subgraph, which is isomorphic to $K_{5,5}$ in Figure 2. Since $C$ is nonreduced, then, $D \subseteq V(\overline{ZD}(S))$. Hence, $\gamma(\overline{ZD}(S)) \geq 3$.

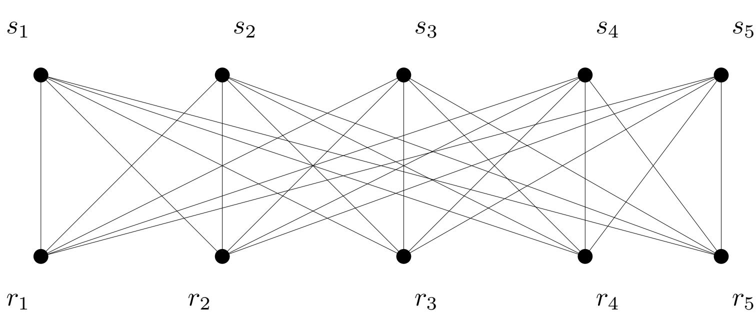


Figure 2: Subgraph of $\overline{ZD}(S)$.

$\square$

**Theorem 2.3.** *Let $S = C \times C \times \cdots \times C$ ($k$ times), where $2 \leq k < \infty$. Then, $\gamma(\overline{ZD}(S)) = 1$, if and only if,*

$$S \cong \frac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle} \times \frac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle}, \quad \text{or } \mathbb{Z}_5 \times \mathbb{Z}_5.$$

*Proof.* If $k \geq 4$, then by Lemma 2.4, we get $\gamma(\overline{ZD}(S)) \geq 3$. If $k = 3$ and $|C| \geq 3$, then by Lemma 2.5, $\gamma(\overline{ZD}(S)) \geq 3$. Also, if $S \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, then from Theorem 2.1, $\overline{ZD}(S)$ is planar. Hence, $k = 2$.

Now, if $C$ is nonreduced ring, then from Lemma 2.6, $\gamma(\overline{ZD}(S)) \geq 3$, not possible by assumption. Let $k = 2$ and $C$ is reduced (not a field). Then, $|C| \geq 6$. Since $\overline{ZD}(S)$ has an induced subgraph $K_{5,5}$, then by Lemma 2.1, $\gamma(\overline{ZD}(S)) \geq 3$, not possible by assumption. Now, we check the rings; $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\frac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle} \times \frac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle}$ and $\mathbb{Z}_5 \times \mathbb{Z}_5$. If we consider the rings $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$, then from Theorem 2.1, $\overline{ZD}(S)$ is a planar, a contradiction. So, the possible rings $S$ for which $\overline{ZD}(S)$ is of genus one are $\frac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle} \times \frac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle}$ or $\mathbb{Z}_5 \times \mathbb{Z}_5$.

Conversely, if $S \cong \frac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle} \times \frac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle}$ or $\mathbb{Z}_5 \times \mathbb{Z}_5$. Then, the graph obtained from $\frac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle} \times \frac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle}$ and $\mathbb{Z}_5 \times \mathbb{Z}_5$ are isomorphic to $K_{3,3}$ and $K_{4,4}$, respectively. Therefore, from Lemma 2.1, we have $\gamma(\overline{ZD}(S)) = 1$. $\square$

**Theorem 2.4.** *Let $S = C \times C \times \cdots \times C$ ($k$ times), where $2 \leq k < \infty$. Then, $\gamma(\overline{ZD}(S)) = 0$ or $1$ or $\geq 3$.*

*Proof.* Assume that $\gamma(\overline{ZD}(S)) = 2$. If we consider $k \geq 4$, then from Lemma 2.4, we get $\gamma(\overline{ZD}(S)) \geq 3$, a contradiction. Hence, $k \leq 3$.

**Case (a)** $k = 3$.
If $|C| \geq 3$, then from Lemma 2.5, $\gamma(\overline{ZD}(S)) \geq 3$, a contradiction. Hence, $|C| = 2$. Also, if $S \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, then from Theorem 2.1, $\overline{ZD}(S)$ is planar.

**Case (b)** $k = 2$.
If $|C| \geq 6$, then, clearly $K_{5,5}$ is a subgraph of $\overline{ZD}(S)$. Therefore, from Lemma 2.1, $\overline{\gamma}(\overline{ZD}(S)) \geq 3$, a contradiction. Hence, $|C| \leq 5$. So the following rings are under consideration: $\mathbb{Z}_5 \times \mathbb{Z}_5$, $\frac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle} \times \frac{\mathbb{Z}_2[x]}{\langle t^2+t+1\rangle}$, $\mathbb{Z}_3 \times \mathbb{Z}_3$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$. Thus, from Theorem 2.1 and Theorem 2.3, we obtain that $\overline{ZD}(S)$ is either planar or of genus 1, again a contradiction. Hence, $\gamma(\overline{ZD}(S)) \geq 3$.

$\square$

**Remark 2.1.** *Let $S = C \times C \times \cdots \times C$ ($k$ times), where $C$ is a ring with $1 \neq 0$ and $2 \leq k < \infty$. Then from Theorem 2.4, we conclude that there is no ring $S$ for which $\gamma(\overline{ZD}(S)) = 2$.*

**Theorem 2.5.** *Let $S = C \times C \times \cdots \times C$ ($k$ times), where $2 \leq k < \infty$. Then, the nonzero genus of $\overline{TD}(S)$ is at least 2.*

*Proof.* Since $\overline{TD}(S)$ has an induced subgraph $\overline{ZD}(S)$, then, from Theorem 2.4, the only possibility of rings $S$ such that $\gamma(\overline{TD}(S)) = 2$ are $\frac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle} \times \frac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle}$ or $\mathbb{Z}_5 \times \mathbb{Z}_5$. If we consider $\mathbb{Z}_5 \times \mathbb{Z}_5$, then, it contains a subgraph which have disjoint copies of two $K_{4,4}$ in Figures 3 and 4. Thus from Lemmas 2.1 and 2.3, $\gamma(\overline{TD}(\mathbb{Z}_5 \times \mathbb{Z}_5)) \geq 2$.

Similarly, if we consider $\frac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle} \times \frac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle}$, then, it contain two disjoint copies of $K_{3,3}$. Therefore, from Lemmas 2.1 and 2.3, $\gamma(\overline{TD}(\frac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle} \times \frac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle}) \geq 2$. Hence, genus of $\overline{TD}(S)$ is at least 2.

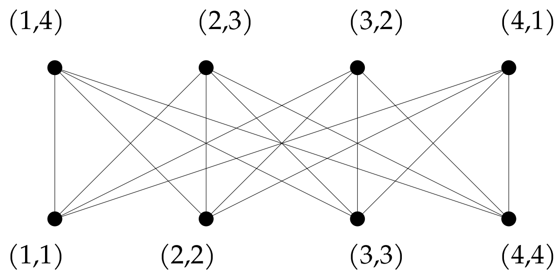


Figure 3: Subgraph of $\overline{TD}(\mathbb{Z}_5 \times \mathbb{Z}_5)$.

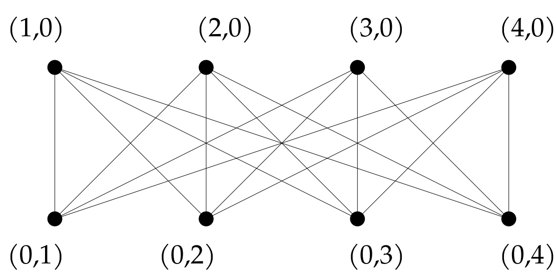


Figure 4: Subgraph of $\overline{TD}(\mathbb{Z}_5 \times \mathbb{Z}_5)$.

$\square$

**Remark 2.2.** *Let $S = C \times C \times \cdots \times C$ ($k$ times), where $2 \leq k < \infty$. Then from Theorem 2.5, we conclude that there is no ring $S$ for which $\gamma(\overline{TD}(S)) = 1$.*

**Theorem 2.6.** *Let $S = C \times C \times \cdots \times C$ ($k$ times), where $2 \leq k < \infty$. Then, $\gamma(\overline{TD}(S)) = 2$ if and only if $S \cong \frac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle} \times \frac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle}$ or $\mathbb{Z}_5 \times \mathbb{Z}_5$.*

*Proof.* Suppose that $k \geq 4$ and $\gamma(\overline{TD}(S)) = 2$. As $\overline{ZD}(S)$ is an induced subgraph of $\overline{TD}(S)$, then from Lemma 2.4 $\gamma(\overline{TD}(S) \geq 3$, a contradiction. Thus, $k \leq 3$. Let, $k \leq 3$. Then from Lemma 2.5, $\gamma(\overline{TD}(S) \geq 3$. Hence, $k = 2$. Assume that $|C| \geq 6$. Then, $K_{5,5}$ is a subgraph of $\overline{TD}(S)$. Therefore, from Lemma 2.1 $\gamma(\overline{TD}(S) \geq 3$. Thus, $|C| \leq 5$. Also, if $S \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$, then from Theorem 2.2, $\overline{TD}(S)$ is planar, which is not possible by assumption. So, the possible rings such that $\gamma(\overline{TD}(S)) = 2$ are $\frac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle} \times \frac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle}$ or $\mathbb{Z}_5 \times \mathbb{Z}_5$.

Conversely, let $S \cong \dfrac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle} \times \dfrac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle}$ or $\mathbb{Z}_5 \times \mathbb{Z}_5$. Then by [5, Theorem 8], $\overline{TD}(S) = TD(S)$. Thus from [14, Theorem 5.2], $\gamma(\overline{TD}(S)) = 2$. □

## 3 Crosscap Characterizations of $TD(S)$ and $ZD(S)$

Let $S = C \times \cdots \times C$ ($k$ times), where $1 \leq k < \infty$. The unit dot product graph $UD(S)$ is defined as the graph whose vertex set is $S^{\times}$, where two distinct vertices $c_1, c_2 \in S^{\times}$ are connected by an edge if and only if $c_1 \cdot c_2 = 0$. The graph $UD(S)$ is a subgraph of $TD(S)$, which is introduced and studied by Abdulla and Badawı in [1]. In this section, we classify the rings $S$ for which $TD(S)$ and $ZD(S)$ are of crosscap one and two.

**Lemma 3.1.** [17, *Theorem 11.23*] $\overline{\gamma}(K_{p,q}) = \left\lceil \dfrac{(p-2)(q-2)}{2} \right\rceil$, *where* $p \geq 1$ *and* $q \geq 1$. *In particular,* $\overline{\gamma}(K_{3,3}) = 1$ *and* $\overline{\gamma}(K_{3,4}) = 1$.

**Corollary 3.1.** [12, *Corollary 1.5*] *Let* $H$ *be a graph with two subgraphs* $H_1$ *and* $H_2$, *where* $H_1$ *and* $H_2$ *are isomorphic to* $K_5$ *or* $K_{3,3}$. *If* $H_1 \cap H_2 = v$, *then,* $\overline{\gamma}(H) \geq 2$.

**Example 3.1.** *Let* $S \cong \dfrac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle} \times \dfrac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle}$. *Then,* $\overline{\gamma}(UD(S)) = 1$.

*Proof.* Here,

$$V(UD(S)) = \Big\{(\overline{1},\overline{1}), (\overline{1},\overline{1+t}), (\overline{t},\overline{t}), (\overline{1+t},\overline{1+t}), (\overline{1+t},\overline{1}), (\overline{t},\overline{1+t}), (\overline{1},\overline{t}), (\overline{t},\overline{1}), (\overline{1+t},\overline{t})\Big\}.$$

Embedding of $UD\left(\dfrac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle} \times \dfrac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle}\right)$ in $\overline{\mathbb{S}}_1$ is shown in the Figure 5.

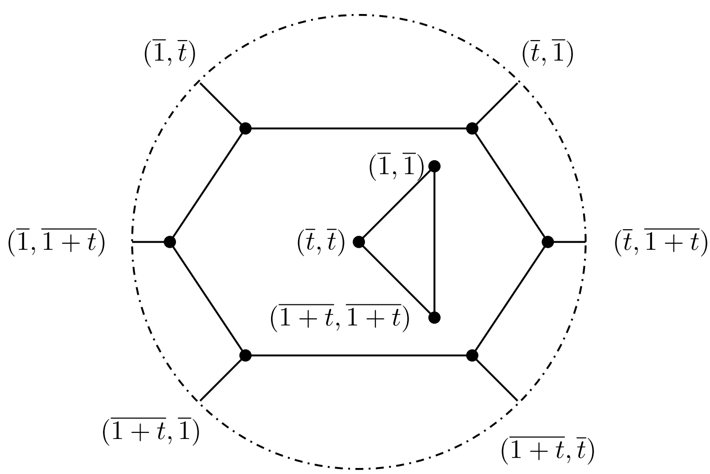


Figure 5: Embedding of $UD\left(\dfrac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle} \times \dfrac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle}\right)$ in $\overline{\mathbb{S}}_1$.

□

**Lemma 3.2.** *If* $C \cong \mathbb{Z}_2$ *and* $S = C \times C \times \cdots \times C$ ($k \geq 4$ *times*), *then,* $\overline{\gamma}(ZD(S)) \geq 3$.

*Proof.* Let us consider the set $H = H_1 \cup H_2$, where

$$H_1 = \Big\{\ell_1 = (1,1,0,0,\dots,0), \ell_2 = (0,0,1,1,0,\dots,0), \ell_3 = (0,1,0,1,0,\dots,0),$$
$$\ell_4 = (1,0,1,0,\dots,0), \ell_5 = (0,1,1,0,\dots,0), \ell_6 = (1,0,0,1,0,\dots,0)\Big\}, \quad \text{and}$$
$$H_2 = \Big\{t_1 = (1,0,0,0,\dots,0), t_2 = (0,0,0,1,0,\dots,0), t_3 = (0,0,1,0,\dots,0),$$
$$t_4 = (1,1,1,0,\dots,0), t_5 = (0,1,1,1,0,\dots,0), t_6 = (1,1,0,1,0,\dots,0),$$
$$t_7 = (1,0,1,1,0,\dots,0)\Big\}.$$

Then, $H \subseteq V(ZD(S))$. On contracting the edges $[\ell_1, \ell_2]$, $[\ell_3, \ell_4]$ and $[\ell_5, \ell_6]$, then, the graph obtained from the set $H$ contains an induced subgraph $K_{3,7}$ in Figure 6. Therefore, from Lemma 3.1, $\overline{\gamma}(ZD(S)) \geq 3$.

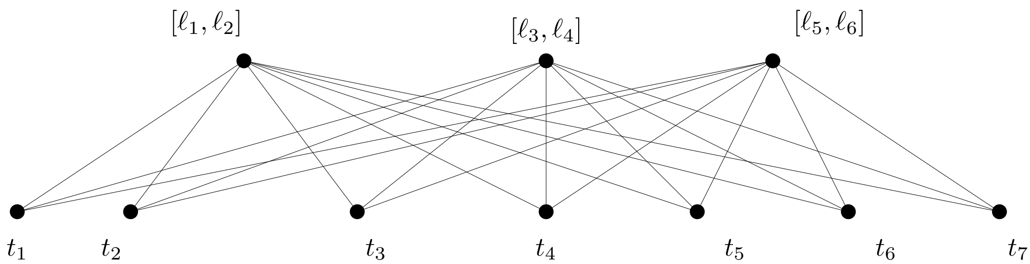


Figure 6: Minor subgraph of $ZD(S)$.

□

**Example 3.2.** *Let $C \cong \mathbb{Z}_3$ and $S = C \times C \times C$. Then, $\overline{\gamma}(ZD(S)) \geq 2$.*

*Proof.* Let,

$$I_1 = \Big\{(1,2,0),(0,0,1),(0,1,0),(2,1,0),(0,0,2),(1,1,0),(2,2,0)\Big\}, \quad \text{and}$$
$$I_2 = \Big\{(1,0,0),(0,1,2),(0,2,1),(0,1,1),(2,0,0),(0,2,0),(0,2,2)\Big\}.$$

Then, $I_1 \cup I_2 \subset V(ZD(S))$. On contracting the edges $\big[(0,0,1),(0,1,0)\big]$, $\big[(2,0,0),(0,2,0)\big]$ and $\big[(0,0,2),(1,0,0)\big]$, we see that the graphs obtained from $I_1$ and $I_2$ are $K_{3,3}$ (shown in Figure 7). Let the graph obtained by set $I_1$ is $H_1$ and the graph obtained by set $I_2$ is $H_2$ Then, clearly $H_1 \cap H_2 = \{[(0,0,2),(1,0,0)]\}$. Thus, the graph in the Figure 7 satisfies the condition of Corollary 3.1. Therefore, $\overline{\gamma}(ZD(S)) \geq 2$.

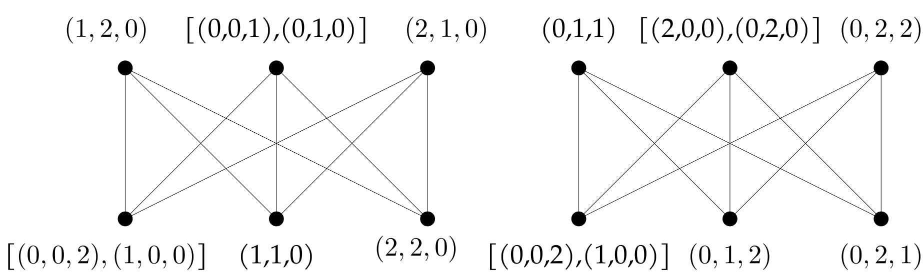


Figure 7: Minor subgraphs of $ZD(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)$.

□

**Theorem 3.1.** *Let $S = C \times C \times \cdots \times C$ ($k$ times), where $2 \leq k < \infty$. Then, $\overline{\gamma}(ZD(S)) = 1$ if and only if $S \cong \frac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle} \times \frac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle}$.*

*Proof.* Suppose that $\overline{\gamma}(ZD(S)) = 1$. Assume $k \geq 4$. Then from Lemma 3.2, $\overline{\gamma}(ZD(S)) \geq 3$, not possible by assumption. Thus, $k \leq 3$. If $k = 3$ and $|C| \geq 3$, then, $ZD(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)$ is a subgraph of $ZD(S)$. Therefore, from Example 3.2, $\overline{\gamma}(ZD(S)) \geq 2$, not possible by assumption. Hence, $k = 2$.

Now, assume that $|C| \geq 5$, then, $K_{4,4}$ is a subgraph of $ZD(S)$. Therefore, from Lemma 3.2, $\overline{\gamma}(ZD(S)) \geq 2$, not possible by assumption. So, possible rings under consideration are $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $\frac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle} \times \frac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle}$. If $S \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, then, $ZD(S)$ is planar by [14, Corollary 3.5]. Thus, $S \cong \frac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle} \times \frac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle}$.

Conversely, if $S \cong \frac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle} \times \frac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle}$, then, $ZD(S) \cong K_{3,3}$. Hence, from Lemma 3.1, $\overline{\gamma}(ZD(S)) = 1$. □

**Theorem 3.2.** *Let $S = C \times C \times \cdots \times C$ ($k$ times), where $2 \leq k < \infty$. Then, $\overline{\gamma}(ZD(S)) = 2$ if and only if, $S \cong \frac{\mathbb{Z}_2[t]}{\langle t^2\rangle} \times \frac{\mathbb{Z}_2[t]}{\langle t^2\rangle}$ or $\mathbb{Z}_4 \times \mathbb{Z}_4$ or $\mathbb{Z}_5 \times \mathbb{Z}_5$.*

*Proof.* Suppose that $\overline{\gamma}(ZD(S)) = 2$. Assume $k \geq 4$. Then from Lemma 3.2, $\overline{\gamma}(ZD(S)) \geq 3$, not possible by assumption. Thus, $k \leq 3$.

Let $k = 3$ and $C \cong \mathbb{Z}_3$, then from Example 3.2, $\overline{\gamma}(ZD(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)) \geq 2$. Claim $\overline{\gamma}\big(ZD(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)\big) \geq 3$. Since $\big\{(1,0,1),(1,0,2),(2,0,2),(2,0,1)\big\} \subset V\big(ZD(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)\big)$. Clearly $(1,0,1)-(1,0,2)-(2,0,2)-(2,0,1)-(1,0,1)$ forms a cycle. Therefore, these vertices lie on the same face during the imbedding into $N_2$. Now, if $(0,1,0)$ and $(0,2,0)$ lie on the distinct copies of $K_{3,3}$, then, there is edge crossing in the embedding.

Also, if $(0,1,0)$ and $(0,2,0)$ lie on the same copies of $K_{3,3}$, then, there is an edge crossing, again a contradiction. Hence, $\overline{\gamma}(ZD(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)) \geq 3$. Now, $ZD(\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3)$ is a subgraph of $ZD(C \times C \times C)$. Therefore, $\overline{\gamma}(ZD(C \times C \times C)) \geq 3$.

Assume that $k = 2$. If $|C| \geq 6$, then, clearly $K_{5,5}$ is a subgraph of $ZD(C \times C)$, from Lemma 3.1, a contradiction. Hence, $|C| \leq 5$. Thus, the following rings under consideration are $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\frac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle} \times \frac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle}$, $\frac{\mathbb{Z}_2[t]}{\langle t^2\rangle} \times \frac{\mathbb{Z}_2[t]}{\langle t^2\rangle}$, $\mathbb{Z}_4 \times \mathbb{Z}_4$ and $\mathbb{Z}_5 \times Z_5$. If $S \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$, then, by [14, Corollary 3.5], $ZD(S)$ is planar, leading to a contradiction. Now, if $S \cong \frac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle} \times \frac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle}$, then from Theorem 3.1, $ZD(S)$ is of crosscap one, again a contradiction. Hence, the possible rings such that $ZD(S)$ is of crosscap two are $\frac{\mathbb{Z}_2[t]}{\langle t^2\rangle} \times \frac{\mathbb{Z}_2[t]}{\langle t^2\rangle}$ or $\mathbb{Z}_4 \times \mathbb{Z}_4$ or $\mathbb{Z}_5 \times \mathbb{Z}_5$.

Conversely, if $S \cong \frac{\mathbb{Z}_2[t]}{\langle t^2\rangle} \times \frac{\mathbb{Z}_2[t]}{\langle x^2\rangle}$ or $\mathbb{Z}_4 \times \mathbb{Z}_4$, then, the embedding of $ZD(S)$ is shown in the

Figure 8. Also, if $S$ is isomorphic to $\mathbb{Z}_5 \times \mathbb{Z}_5$, then, $ZD(S) \cong K_{4,4}$. Therefore, from Lemma 3.1, $\overline{\gamma}(ZD(\mathbb{Z}_5 \times \mathbb{Z}_5)) = 2$.

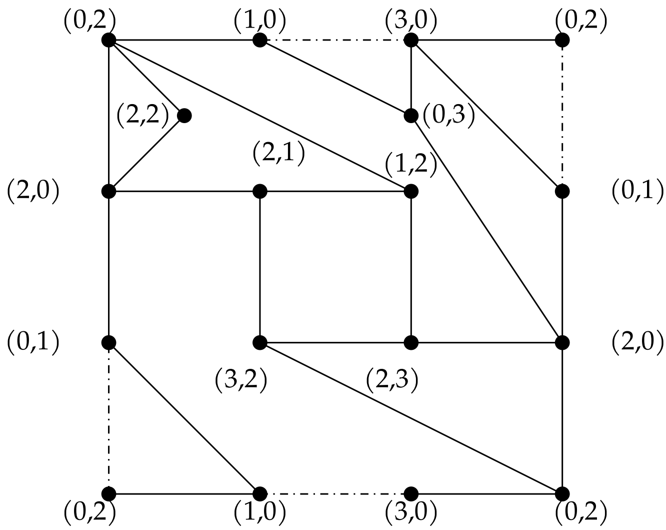


Figure 8: Embedding of $ZD(\mathbb{Z}_4 \times \mathbb{Z}_4) \cong ZD\left(\frac{\mathbb{Z}_2[t]}{\langle t^2\rangle} \times \frac{\mathbb{Z}_2[t]}{\langle t^2\rangle}\right)$ in $\overline{\mathbb{S}}_2$.

$\square$

**Remark 3.1.** *If $H$ is a disconnected graph with connected components $H_1$, $H_2$,...,$H_k$ such that $\bigcap_{i=1}^{k} H_i = \phi$ and $\overline{\gamma}(H_i)$ is the non-orientable genus of $i-th$ connected component, then, $\overline{\gamma}(H) = \sum_{1}^{k} \overline{\gamma}(H_i)$. This relationship holds due to the additive nature of the non-orientable genus when dealing with disconnected graphs. So, we use this concept in next result whenever graph is disconnected.*

**Theorem 3.3.** *Let $S = C \times C \times \cdots \times C$ ($k$ times), where $2 \leq k < \infty$. Then, $\overline{\gamma}(TD(S)) = 2$ if and only if, $S \cong \frac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle} \times \frac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle}$ or $\frac{\mathbb{Z}_2[t]}{\langle t^2\rangle} \times \frac{\mathbb{Z}_2[t]}{\langle t^2\rangle}$ or $\mathbb{Z}_4 \times \mathbb{Z}_4$.*

*Proof.* Since $TD(S)$ contains $ZD(S)$, then, from Theorems 3.1 and 3.2, the only possibility of $TD(S)$ is of crosscap two are $\frac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle} \times \frac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle}$ or $\frac{\mathbb{Z}_2[t]}{\langle t^2\rangle} \times \frac{\mathbb{Z}_2[t]}{\langle t^2\rangle}$ or $\mathbb{Z}_4 \times \mathbb{Z}_4$ or $\mathbb{Z}_5 \times \mathbb{Z}_5$. Let $S \cong \mathbb{Z}_5 \times \mathbb{Z}_5$. Then from Theorem 3.2, $\overline{\gamma}(ZD(S)) = 2$. Also $UD(S)$ has a subgraph $K_{4,4}$, therefore from Lemma 3.1, $\overline{\gamma}(UD(S)) \geq 2$. Since $ZD(S)$ and $UD(S)$ have total three connected components of $TD(S)$ with $V(ZD(S)) \cap V(UD(S)) = \phi$ and $E(ZD(S)) \cap E(UD(S)) = \phi$.

Therefore, due to additive nature of crosscap, we have $\overline{\gamma}(TD(S)) \geq 4$, a contradiction. Thus, $S \cong \frac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle} \times \frac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle}$ or $\frac{\mathbb{Z}_2[t]}{\langle t^2\rangle} \times \frac{\mathbb{Z}_2[t]}{\langle t^2\rangle}$ or $\mathbb{Z}_4 \times \mathbb{Z}_4$.

Conversely, if $S \cong \frac{\mathbb{Z}_2[t]}{\langle t^2\rangle} \times \frac{\mathbb{Z}_2[t]}{\langle t^2\rangle}$ or $\mathbb{Z}_4 \times \mathbb{Z}_4$. Then, the embedding of $TD(S)$ is shown in the

Figure 9. Next, if $S \cong \dfrac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle} \times \dfrac{\mathbb{Z}_2[t]}{\langle t^2+t+1\rangle}$, then, $TD(S)$ is disconnected graph. Since $TD(S) = ZD(S) \cup UD(S)$ with $E(ZD(S)) \cap E(UD(S)) = \phi$ and $V(ZD(S)) \cap V(UD(S)) = \phi$. Thus, $\overline{\gamma}(TD(S)) = \overline{\gamma}(ZD(S)) + \overline{\gamma}(UD(S))$. From Theorem 3.1, $\overline{\gamma}(ZD(S)) = 1$ and from Example 3.1, $\overline{\gamma}(UD(S)) = 1$. Therefore, due to additive nature of the non-orientable genus, we have $\overline{\gamma}(TD(S)) = 2$.

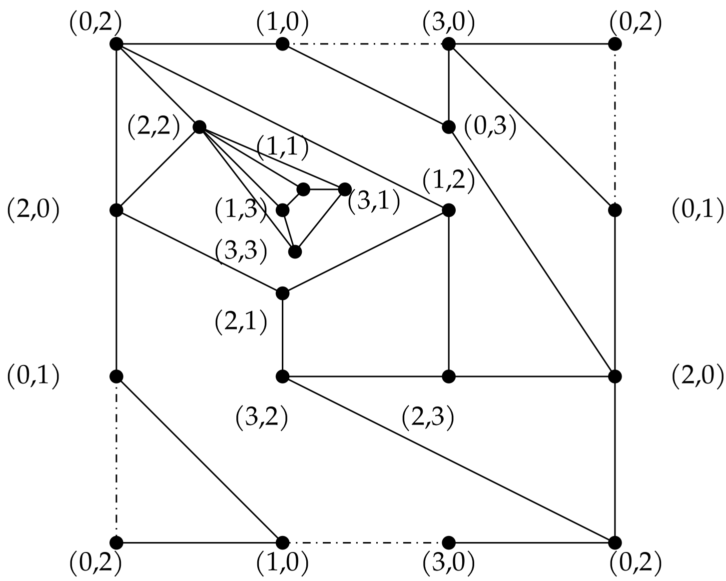


Figure 9: Embedding of $TD(\mathbb{Z}_4 \times \mathbb{Z}_4) \cong TD\left(\frac{\mathbb{Z}_2[t]}{\langle t^2\rangle} \times \frac{\mathbb{Z}_2[t]}{\langle t^2\rangle}\right)$ in $\overline{\mathbb{S}}_2$.

$\square$

## 4 Conclusion

Let $S = C \times C \times \cdots \times C$ (k times), where $2 \leq k < \infty$. In this work, we first consider the graphs $\overline{ZD}(S)$ and $\overline{TD}(S)$ introduced by Asma et al. [5] and categorize the commutative rings that exhibit toroidal and bi-toroidal graphs. We also examine the graphs $ZD(S)$ and $TD(S)$ introduced by Badawi, and obtain the rings $S$ for which these graphs have crosscap numbers one and two.

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**Conflicts of Interest** The authors declare no conflict of interest.

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